Minimum Induced Drag of a Hemi-Circular Ground Effect Wing

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The induced drag is considered for a hemi-circular front view wing with both tips in close proximity to the ground surface. The integral equation is exactly solved using Söhngen's inversion formula. Assuming an optimum downwash distribution, the exact expression of the span-efficiency-factor is discussed analytically and numerically.

Nomenclature

D, $D_{\min}$	= Induced drag, the minimum induced drag.
$K(\nu),E(\nu)$	= complete elliptic integral of the first and the second kinds
e_0	= span-efficiency-parameter
K	= the maximum value of the normal downwash $w(\theta)$
L	= total lift
U	= freestream velocity
$w(\theta) = w(\mathrm{x})$	= normal downwash
$\chi = \cos\theta$	= nondimensional spanwise coordinate
$\Gamma(\mathrm{x})$	= strength of the bound vortex
$\gamma(\mathrm{x})$	= strength of the trailing vortex sheet
ϵ	= the θ-value which determines the wing tip gap.
ξ	= nondimensional variable
θ	= angular variable which defines a point on a hemi-circular arc
ν	= $\cos\epsilon$
ρ	= air density

Remarks

Let the induced drag coefficient and the lift coefficient be $C_D = D/\frac{1}{2}\rho U^2(2R\nu)c$ and $C_L = L/\frac{1}{2}\rho U^2(2R\nu)c$ respectively, where c = wing chord length and R = radius of the front view. Then we have the familiar expression $C_D = C_L^2/\pi Æ e_0$, where $Æ = 2R\nu/c$.

Introduction

THE induced drag of the GEW (Ground Effect Wing), was first studied by S. Ando[1] for a hemi-circular front view wing. He obtained the series expansions for the span-efficiency-parameter, the spanwise lift distribution etc. Next, he[2] tried to extend his theory to the hemi-elliptic front view wing. He also presented the series expansions of the spanwise lift distribution, the total lift, and the span-efficiency-parameter.

Thereafter, P. A. Ashill presented his theory on the minimum induced drag of the GEW which had the ⊓ shaped front view. Because of the rather sophisticated results, however, the limiting characteristics for the vanishing wing tip gap were not explicitly shown. The present authors believe that these limiting characteristics are the most important ones from a theoretical standpoint.

In the present paper, the exact solution for the hemi-circular front view GEW is presented, using Söhngen's inversion formula, and the behaviour for small gaps is clearly shown.

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Integral Equation and Total Lift

We assume as in usual theories that the trailing vortex sheet extends infinitely downstream, keeping the same front view as that of the wing. Since the induced drag is determined by only the flow pattern in the Trefftz-plane, it is valid to make use of the lifting line theory. Fig. 1 shows that a hemi-circular front view wing is in proximity to the ground surface, together with some relevant notations.

Let the strength of the trailing vortex sheet per unit arc-length be $\gamma(\theta)$. Then the normal downwash $w(\theta)$ on the actual wing is expressed in terms of $\gamma(\theta)$[1]

$$w(\theta) = \frac{1}{4\pi}\int_{\epsilon}^{\pi-\epsilon}\gamma(\theta_1)\frac{\sin\theta_1}{\cos\theta_1 - \cos\theta}d\theta_1 \tag{1}$$

Introducing a variable $\mathrm{x} = \cos\theta$, we have from Eq. (1)

$$w(x) = \frac{1}{4\pi}\int_{-\nu}^{\nu}\gamma(x_1)\frac{dx_1}{x_1 - x} \tag{2}$$

where $\nu = \cos\theta$.

Equation (1) is regarded as an integral equation for $\gamma(\mathrm{x})$, when $w(\mathrm{x})$ is known. Söhngen's inversion formula gives the general solution as follows:

$$\sqrt{\nu^2 - x^2}\cdot\gamma(x) = -\frac{4}{\pi}\int_{-\nu}^{\nu}\frac{\sqrt{\nu^2-\xi^2}}{\xi - x}w(\xi)d\xi + \text{const.} \tag{3}$$

We concentrate our interest on the case of the symmetric airload throughout this paper. Then we have

$$w(x) = w(-x) \quad \text{and} \quad \gamma(x) = -\gamma(-x)$$

which make the additional constant in Eq. (3) vanish.

The strength of the bound vortex $\Gamma(\theta)$ at a station θ should be equal to the total trailing vortices starting between this point θ and the closer wing tip. Then we have

$$\Gamma(\theta) = \Gamma(x) = \int_{\epsilon}^{\theta}\gamma(\theta_1)Rd\theta_1 = \int_{x}^{\nu}R\frac{\gamma(x_1)}{\sqrt{1-x_1^2}}dx_1 \tag{4}$$

Using Eq. (4), we also have the expressions for the total lift and the induced drag

$$\left.\begin{aligned} L &= \int_{\epsilon}^{\pi-\epsilon}\rho U\Gamma(\theta)\sin\theta d\theta = \rho UR\int_{-\nu}^{\nu}\Gamma(x)dx \\ D &= \int_{\epsilon}^{\pi-\epsilon}\rho w(\theta)\Gamma(\theta)d\theta = \rho R\int_{-\nu}^{\nu}w(x)\Gamma(x)\frac{dx}{\sqrt{1-x^2}} \end{aligned}\right\} \tag{5}$$

Substitution of Eq. (4) into Eq. (5) leads to

$$\begin{aligned} L &= \rho UR^2\int_{-\nu}^{\nu}dx\int_{x}^{\nu}\gamma(x_1)\frac{dx_1}{\sqrt{1-x_1^2}} \\ &= \rho UR^2\int_{-\nu}^{\nu}\frac{\nu + x_1}{\sqrt{1-x_1^2}}\gamma(x_1)dx_1 \end{aligned} \tag{6}$$

Then substituting Eq. (3) with const = 0 into Eqs. (4) and

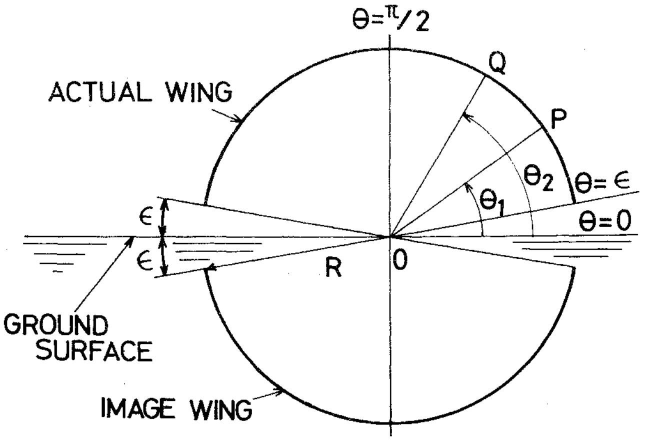

Fig. 1 Hemi-circular front view GEW, and some relevant notations.

(6), we have

$$\Gamma(x) = -\frac{4R}{\pi}\int_{-\nu}^{\nu} d\xi \int_{x}^{\nu} dx_1 \frac{w(\xi)\sqrt{\nu^2-\xi^2}}{\xi - x}\,\frac{1}{\sqrt{1-x_1{}^2}\sqrt{\nu^2 - x_1{}^2}} \tag{7}$$

and

$$\mathrm{L} = -\frac{4\rho U}{\pi}R^2\int_{-\nu}^{\nu}\int_{-\nu}^{\nu}\frac{w(\xi)\sqrt{\nu^2-\xi^2}}{\xi - x}\,\frac{\nu + x}{\sqrt{1-x^2}\sqrt{\nu^2-x^2}}d\xi dx \tag{8}$$

both of which are double integrals containing $w(\xi)$.

The Case of Optimum Lift Distribution.

Spanwise lift distribution and total lift

For the optimum lift distribution, Ando[1,2] gives the normal downwash as follows:

$$w(\theta) = w(x) = K\sin\theta = K\sqrt{1-x^2} \tag{9}$$

Substitution of Eq. (9) into Eq. (5) proves the important relation

$$UD_{\min} = KL \tag{10}$$

immediately. Substitution of Eq. (9) into Eq. (8) gives the total lift as follows:

$$\mathrm{L} = -\frac{4\rho UR^2K}{\pi}\int_{-\nu}^{\nu}\int_{-\nu}^{\nu}d\xi dx\frac{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}}{\xi - x}\,\frac{\nu + x}{\sqrt{1-x^2}\sqrt{\nu^2-x^2}} \tag{11}$$

Define the Eq (11′) by exchanging ξ for x in Eq. (1), and add the Eq (11′) to Eq. (11). Then we have

$$L = \frac{2\rho UR^2K}{\pi}\int_{-\nu}^{\nu}\int_{-\nu}^{\nu}d\xi dx\,\frac{1}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}\sqrt{1-x^2}\sqrt{\nu^2-x^2}} \times \frac{1}{\xi - x} \tag{12}$$

$$\times\{(1-x^2)(\nu^2-x^2)(\nu+\xi) - (1-\xi^2)(\nu^2-\xi^2)(\nu+x)\}$$

The expression in the above bracket reduces to

$$\{\ \} = (\xi - x)[\nu(\nu^2-1)(\xi+x) - \nu(\xi+x)(\xi^2+x^2) + \nu^2 + (\nu^2+1)\xi x - \xi x(\xi^2+\xi x+x^2)]$$

Taking account of the symmetry of the integrals and picking up the nonzero terms only, we have from Eq. (12)

$$L = \frac{2\rho UR^2K}{\pi}\int_{-\nu}^{\nu}\int_{-\nu}^{\nu}d\xi dx\,\frac{\nu^2 - x^2\xi^2}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}\sqrt{1-x^2}\sqrt{\nu^2-x^2}} \tag{13}$$

Here make use of

$$\int_0^{\nu}\frac{d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}} = K(\nu)$$

and

$$\int_0^{\nu}\frac{\xi^2 d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}} = \int_0^{\nu}\frac{d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}} - \int_0^{\nu}\frac{\sqrt{1-\xi^2}}{\sqrt{\nu^2-\xi^2}}d\xi = K(\nu) - E(\nu)$$

Then the total lift is given in the following expression:

$$L = \frac{8}{\pi}\rho UK\cdot(R\nu)^2\left[K^2(\nu) - \frac{1}{\nu^2}\{K(\nu) - E(\nu)\}^2\right] \tag{14}$$

Also, from Eqs. (7) and (9), we obtain the lift distribution Γ (x) as follows:

$$\left[\frac{\Gamma(x)}{\Gamma(x=0)}\right] = \frac{\sqrt{\nu^2-x^2}}{\nu K(\nu)}\left\{\frac{\pi}{2} + \int_0^{\nu}\frac{\xi^2}{\sqrt{1-x^2}+\sqrt{1-\xi^2}}\,\frac{d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}}\right\} \tag{15}$$

For the vanishing wing tip gaps (Eq. $\nu \simeq 1$, or $\epsilon \gg 1$), we have the asymptotic form of Eq. (15) as follows:

$$\frac{\Gamma(x)}{\Gamma(0)} \simeq \frac{\sqrt{\nu^2-x^2}}{\nu K(\nu)} \times \frac{K(\nu)}{\sqrt{1-x^2}} \rightarrow 1, \quad \text{for} \quad \nu \rightarrow 1 \tag{16}$$

which shows that the spanwise lift distribution becomes uniform, as $\nu \rightarrow 1$ (or $\epsilon \rightarrow 0$). For another limiting case of $\nu \sim 0$ (or $\epsilon \sim \pi/2$), we obtain

$$\Gamma(x)/\Gamma(0) \simeq \sqrt{\nu^2-x^2}\left\{1 + \frac{1}{4}\nu^2 + \ldots\ldots\ldots\right\} \tag{17}$$

which shows that the spanwise lift distribution becomes elliptic as $\nu \rightarrow 0$. Fig. 2 shows the numerical results for $\Gamma(x)/\Gamma(0)$. Decrease in ϵ makes the spanwise lift distribution more and more uniform, but the variation is quite slow.

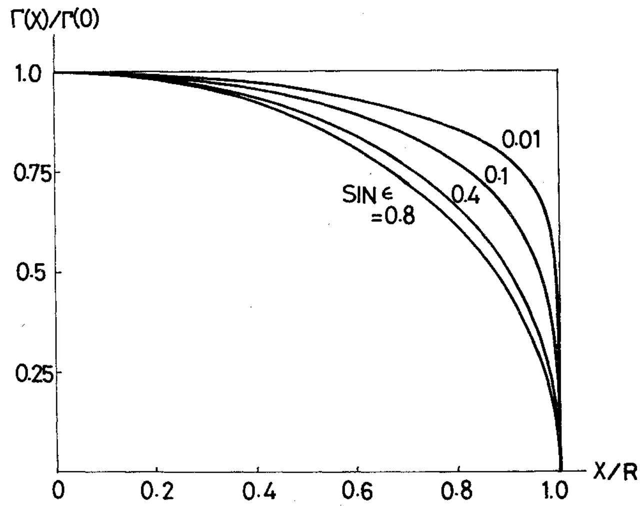

Fig. 2 Variation of spanwise lift distribution with the tip gap parameter ε.

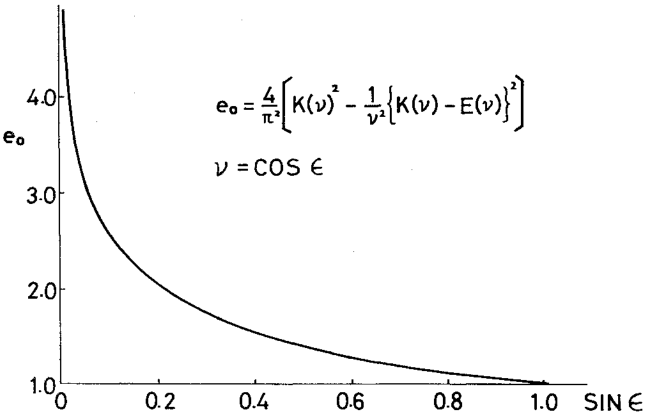

Fig. 3 Span-efficiency-parameter vs tip gap parameter; exact solution.

Span-Efficiency-Parameter

The span-efficiency-parameter e_0 for the hemi-circular front-view wing is defined as

$$e_0 \equiv L^2 / \left[\frac{1}{2}\rho U^2 \pi (2R\nu)^2 D\right] \tag{18}$$

Substitution of Eqs. (10) and (14) into Eq. (18) gives

$$e_0 = L/[2\pi\rho UK\cdot(R\nu)^2]$$
$$= \frac{4}{\pi^2}\left[K^2(\nu) - \frac{1}{\nu^2}\{K(\nu) - E(\nu)\}^2\right] \tag{19}$$

This expression Eq. (19), which is exact, is very much simpler than Ashill's results.[3]

When both wing tips are very close to the ground surface, namely $\nu = \cos\epsilon \sim 1$ ($\epsilon \ll 1$), expansion formulas of the complete elliptic integrals lead us to

$$e_0 = \frac{8}{\pi^2}\left[\log\frac{1}{\sin\epsilon} + 0.88629\right] \tag{20}$$

which is just the result derived by Ando[1] starting with series expansion of $\gamma(x)$. The terms neglected in Eq. (20) are found to be

$$0\left(\sin^{2n}\epsilon\cdot\log^2\frac{1}{\sin\epsilon}\right), \quad 0\left(\sin^{2n}\epsilon\cdot\log\frac{1}{\sin\epsilon}\right)$$

and $0(\sin^{2n}\epsilon)$, for $n \geqq 1$. Figs. 3 and 4 show e_0 vs $\sin\epsilon$. It is found that when $\sin\epsilon \lesssim 0.1$, the value of e_0 is approximated satisfactorily by Eq. (20).

Conclusions

The induced drag of the hemi-circular front view Ground Effect Wing is discussed, particularly in the case of the optimum lift distribution. This hemi-circular model is of interest because the exact development of the theory can be accomplished using Söhngen's inversion formula.

A simple exact formula is derived for the span efficiency parameter, which is simpler than the result given previously by Ashill,[3] although the front view shapes considered by the two theories are somewhat different.

Previous approximate results by S. Ando[1] are confirmed for the theoretical limiting characteristics, such as

$$v_g \simeq 0(1/\epsilon \log\epsilon) \to \infty$$
$$D \simeq 0(1/\log\epsilon) \to 0$$

where v_g denotes the fluid velocity passing through the wing tip gap.

Appendix Calculation of the Lift Distribution

From Eqs. (7) and (9), we obtain

$$\Gamma(x) = -\frac{4RK}{\pi}\int_{-\nu}^{\nu} d\xi \int_{x}^{\nu} dx_1 \frac{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}}{\xi - x_1} \frac{1}{\sqrt{1-x_1^2}\sqrt{\nu^2-x_1^2}} \tag{A1}$$

Now, we have the following relation

$$\int_{-\nu}^{\nu} \frac{\sqrt{\nu^2-\xi^2}}{\xi - x_1}\sqrt{1-\xi^2}\, d\xi$$
$$= \sqrt{1-x_1^2}\int_{-\nu}^{\nu}\frac{\sqrt{\nu^2-\xi^2}}{\xi - x_1} d\xi + \int_{-\nu}^{\nu}\frac{\sqrt{1-\xi^2}-\sqrt{1-x_1^2}}{\xi - x_1}\sqrt{\nu^2-\xi^2}\, d\xi$$
$$= -\pi x_1\sqrt{1-x_1^2} - \int_{-\nu}^{\nu}\frac{\xi + x_1}{\sqrt{1-\xi^2}+\sqrt{1-x_1^2}}\sqrt{\nu^2-\xi^2}\, d\xi$$
$$= -\pi x_1\sqrt{1-x_1^2} - x_1\int_{-\nu}^{\nu}\frac{\sqrt{\nu^2-\xi^2}}{\sqrt{1-x_1^2}+\sqrt{1-\xi^2}} d\xi \tag{A2}$$

Substitution of Eqs. (A2) into Eq. (A1) gives

$$\Gamma(x) = 4KR\int_{x}^{\nu}\frac{x_1}{\sqrt{\nu^2-x_1^2}} dx_1 + \frac{4}{\pi}KR\int_{-\nu}^{\nu} d\xi\sqrt{\nu^2-\xi^2}\int_{x}^{\nu}\frac{x_1 dx_1}{\sqrt{\nu^2-x_1^2}\sqrt{1-x_1^2}(\sqrt{1-x_1^2}+\sqrt{1-\xi^2})} \tag{A3}$$

The first integral is easily carried out to give $\sqrt{\nu^2 - x^2}$. The x_1-integral of the second term reduces to

$$\int_{\sqrt{1-\nu^2}}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{\nu^2 - 1 + y^2}(y + \sqrt{1-\xi^2})}$$

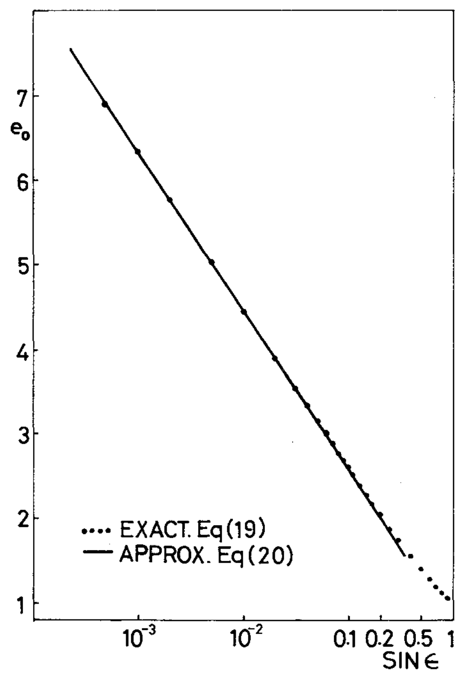

Fig. 4 Span-efficiency-parameter vs tip gap parameter; exact and approximate solutions.

$$= \frac{1}{\sqrt{\nu^2 - \xi^2}} \log \left| \frac{y + \sqrt{1-\xi^2}}{2(\nu^2 - \xi^2) - 2(y + \sqrt{1-\xi^2})\sqrt{1-\xi^2} + 2\sqrt{\nu^2-\xi^2}\sqrt{\nu^2 - 1 + y^2}} \right|_{\sqrt{1-\nu^2}}^{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{\nu^2 - \xi^2}} \log \left| \frac{\sqrt{1-\nu^2}(\sqrt{1-x^2} + \sqrt{1-\xi^2})}{\sqrt{1-x^2}\sqrt{1-\xi^2} - \sqrt{\nu^2 - x^2}\sqrt{\nu^2-\xi^2} + (1-\nu^2)} \right|$$

Then the second term of Eq. (A3) becomes

$$\int_0^\nu dz \log \left| \frac{\cdots\cdots}{\cdots\cdots} \right|$$

$$= \xi \log \left| \frac{\cdots\cdots}{\cdots\cdots} \right| \int_0^\nu + \int_0^\nu dz \frac{1}{(\sqrt{1-x^2} + \sqrt{1-\xi^2})} \frac{\xi^2}{\sqrt{1-\xi^2}}$$

$$- \int_0^\nu dz \frac{\sqrt{1-x^2}\sqrt{\nu^2-\xi^2} - \sqrt{\nu^2-\xi^2}\sqrt{1-\xi^2}}{\sqrt{1-x^2}\sqrt{1-\xi^2} - \sqrt{\nu^2-x^2}\sqrt{\nu^2-\xi^2} + (1-\nu^2)}$$

$$\times \frac{\xi^2}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}}$$

the first term of which vanishes. The second and the third terms are reduced to a common denominator, then the numerator becomes

$$\sqrt{\nu^2 - x^2}\left\{\sqrt{1-x^2}\sqrt{1-\xi^2} - \sqrt{\nu^2-\xi^2}\sqrt{\nu^2-x^2} + (1-\nu^2)\right\}$$

which cancels the denominator partially. Finally we have

$$\Gamma(x) = 4KR\sqrt{\nu^2 - x^2}\left\{1 + \frac{2}{\pi}\int_0^\nu \frac{\xi^2}{\sqrt{1-x^2} + \sqrt{1-\xi^2}} \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}}\right\} \quad \text{(A4)}$$

which, for x = 0, reduces to

$$\Gamma(x=0) = 4KR\cdot\nu \left\{1 + \frac{2}{\pi}\int_0^\nu \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}} - \frac{2}{\pi}\int_0^\nu \frac{d\xi}{\sqrt{\nu^2-\xi^2}}\right\}$$

$$= \frac{8}{\pi} KR\cdot\nu\cdot K(\nu) \quad \text{(A5)}$$

From Eqs. (A4) and (A5), we obtain

$$\frac{\Gamma(x)}{\Gamma(x=0)} = \frac{\sqrt{\nu^2 - x^2}}{\nu K(\nu)} \left\{\frac{\pi}{2} + \int_0^\nu \frac{1}{\sqrt{1-x^2} + \sqrt{1-\xi^2}} \frac{\xi^2}{\sqrt{1-\xi^2}\sqrt{\nu^2-\xi^2}} d\xi\right\} \quad \text{(A6)}$$

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